

# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2364

ON TWO-DIMENSIONAL FLOW AFTER A CURVED STATIONARY  
SHOCK (WITH SPECIAL REFERENCE TO THE PROBLEM  
OF DETACHED SHOCK WAVES)

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## SUMMARY

The problem of two-dimensional flow behind a curved stationary shock wave is considered analytically. The method assumes a given shock-wave shape, which automatically determines certain initial conditions on the flow variables; and the flow pattern, including any body shape, follows from the initial conditions. Approximate analytic expressions are found for the stream function in the subsonic region following the shock and, after the stream function is obtained, the flow density is determined by Bernoulli's equation which connects the density with the derivatives of the stream function. The final solution can then be determined from the velocity field thus obtained.

## INTRODUCTION

The problem of compressible flow after a curved shock wave has been investigated by various authors (references 1 to 4). In the analytical treatments, the main interest has so far been concerned with the local properties of the flow, such as the relations between the gradients of various physical and geometrical quantities along the shock wave and those along the body.

The purpose of the present work is an attempt to treat analytically the two-dimensional problem after the shock in the large. The method assumes a given shock-wave shape, which automatically determines certain initial conditions on the flow variables. It is therefore a Cauchy problem and the flow pattern, including any body profile, follows from the assumed initial conditions. Approximate analytic expressions for the stream function are found for the compressible flow in the whole subsonic region. The stream function thus found satisfies the exact shock conditions but it satisfies the differential equation only approximately. After the stream function is obtained, the flow density is

determined by the exact Bernoulli equation which connects the gas density with the derivatives of the stream function. Then the sonic line can be determined from the velocity field thus obtained.

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#### SYMBOLS

|                             |  |
|-----------------------------|--|
| $\gamma$                    | ratio of specific heats of fluids  |
| $p_1$                       | free-stream pressure   |
| $\rho_1$                    | free-stream density  |
| $U$                         | free-stream velocity magnitude, assumed to be parallel to x-axis   |
| $c_1$                       | free-stream velocity of sound  |
| $F(\rho_1 U y)$             | function determining shock shape (1)   |
| $(x_0, 0)$                  | shock nose when $F(0) = 0$   |
| $\rho_0$                    | stagnation density after shock wave  |
| $c_0$                       | stagnation velocity of sound after shock wave  |
| $g^2 = \psi_x^2 + \psi_y^2$ | where $\psi$ is stream function  |
| $\Omega$                    | complex variable in hodograph plane<br>$\left( (\psi_x - i\psi_y) / \left( \sqrt{\rho_0^2 c_0^2 - g^2} + \rho_0 c_0 \right) \right)$ |
| $\vec{\xi}_0(r)$            | parametric representation of a three-dimensional curve with parameter $r$  |
| $P, Q$                      | complex variables  |

$$x = \frac{1}{\rho_1 U} F(\rho_1 U y) + x_0 \quad \text{equation for shock}$$

|                        |   |
|------------------------|---|
| $M_1$                  | free-stream Mach number $\left( (\gamma p_1 / \rho_1)^{-1/2} U = U / c_1 \right)$ |
| $\psi(x, y)$           | stream function of flow after shock wave $(\rho_1 U y$ on the shock)              |
| $\tan \tau$            | slope of shock wave at point on it  |
| $p$                    | pressure after shock wave   |
| $\rho$                 | density after shock wave  |
| $G(\psi)$              | entropy function  |
| $F(\psi)$              | function determining shock shape  |
| $l, m$                 | constants defined by equations (8)  |
| $u, v$                 | velocity components of flow after shock wave                                      |
| $q = \sqrt{u^2 + v^2}$ |   |
| $c$                    | local velocity of sound after shock wave  |
| $q_M$                  | maximum velocity magnitude of flow  |
| $s, n$                 | tangential and normal directions at point on shock wave                           |
| $q_t, q_n$             | tangential and normal components of velocity of flow immediately after shock wave |
| $K_1, K_2$             | constants defined by equations (18)   |
| $F(y)$                 | analytic function in real variable $y$  |
| $\phi(x, y)$           | auxiliary function defined by equations (22)                                      |
| $h(\Omega)$            | arbitrary analytic function of $\Omega$   |
| $\text{Re}$            | real part of a complex quantity   |

$$k = \rho_1 U / \rho_0 c_0$$

$$z = x + yi$$

$f_1(t)$  analytic function for Weierstrass' parametric representation of a minimal surface

$t$  complex variable occurring in Weierstrass' formula for minimal surfaces

$\text{Im}$  imaginary part of a complex quantity

$\vec{\xi}_0$  vector function defined by equation (29)

$\vec{\xi}$  vector with components  $x$ ,  $y$ , and  $w$

$\vec{\eta}(P), \vec{\xi}(Q)$  complex-valued vector functions of variables  $P$  and  $Q$ , respectively; components denoted by subscripts 1, 2, and 3

$\vec{\kappa}(r)$  unit vector normal to tangential direction of  $\vec{\xi}_0(r)$

$\vec{\pi}(y)$  vector function defined by equation (33)

$\sigma = y + i\omega$  complex variable

$\omega$  imaginary part of complex variable  $\sigma$

#### FORMULATION OF PROBLEM

The present report is concerned with a stationary shock wave produced by a two-dimensional body placed in a uniform supersonic stream. The fluid is assumed to be an ideal gas and the process on any material particle, to be adiabatic. Thus, the entropy is constant along a streamline while it may vary from one streamline to the next.

Let the coordinate axes be so chosen that the incoming free stream is along the  $x$ -axis and that the shape of the shock wave is represented by

$$x = \left[ F(\rho_1 U y) / \rho_1 U \right] + x_0 \quad (1)$$

where  $U$  is the free-stream velocity and  $\rho_1$  is the gas density in the free stream. As the shock wave must asymptotically become the Mach line at infinity, the function  $F'(\rho_1 U y) \rightarrow \pm 1/\sqrt{M_1^2 - 1}$  as  $y \rightarrow \pm\infty$ , where  $M_1$  is the free-stream Mach number.

Entropy of flow after shock waves.— The stream function  $\psi(x,y)$  may be regarded as the amount of the flow across a curve joining the point  $(x,y)$  to the point  $(x_0,0)$ . It must therefore be continuous across the shock wave because of the continuity of the mass. Thus, on the shock wave,

$$\psi(x,y) = \rho_1 U y \quad (2)$$

The slope of the shock wave is therefore

$$\tan \tau = \frac{dy}{dx} = 1/F'(\psi) \quad (3)$$

Since the entropy is constant along a streamline, the pressure-density relation for the flow after the shock wave can be written in the form

$$p/\rho^\gamma = G(\psi)p_1/\rho_1^\gamma \quad (4)$$

where  $\gamma$  is the ratio of the specific heats of the fluid,  $p_1$  and  $p$  are, respectively, the pressures before and after the shock, and  $\rho_1$  and  $\rho$  are corresponding densities of the fluid. This entropy function  $G(\psi)$  is now to be expressed in terms of the function  $F(\psi)$  which determines the shock shape. From the shock conditions, it can be immediately shown that

$$p/p_1 = \frac{2\gamma M_1^2 \sin^2 \tau + (1 - \gamma)}{\gamma + 1} \quad (5)$$

$$\rho/\rho_1 = \frac{(\gamma + 1)M_1^2 \sin^2 \tau}{2 + (\gamma - 1)M_1^2 \sin^2 \tau} \quad (6)$$

Therefore

$$G(\psi) = \frac{1}{(\gamma + 1)^{\gamma+1}} \frac{2\gamma M_1^2 + (1 - \gamma) [1 + (F')^2]}{1 + (F')^2} \left\{ \frac{2}{M_1^2} [1 + (F')^2] + \gamma - 1 \right\}^{\gamma} \quad (7)$$

where  $F' = F'(\psi)$ . Thus one has the formula

$$G'(\psi)/G(\psi) = 2F'(\psi)F''(\psi) \left[ \frac{l}{1 + l(F')^2} + \frac{m\gamma}{1 + m(F')^2} - \frac{1}{1 + (F')^2} \right] \quad (8a)$$

where

$$l = (1 - \gamma) / (2\gamma M_1^2 - \gamma + 1) \quad (8b)$$

$$m = 2 / [2 + (\gamma - 1)M_1^2] \quad (8c)$$

Fundamental differential equation for stream function.- By the definition of the stream function given above,

$$\left. \begin{aligned} \rho u &= \frac{\partial \psi}{\partial y} \\ \rho v &= -\frac{\partial \psi}{\partial x} \end{aligned} \right\} \quad (9)$$

From the equation of vorticity (reference 3)

$$\frac{\partial}{\partial x} \left( \frac{1}{\rho} \psi_x \right) + \frac{\partial}{\partial y} \left( \frac{1}{\rho} \psi_y \right) = -G'(\psi) \rho^\gamma p_1 / (\gamma - 1) \rho_1^\gamma \quad (10)$$

and Bernoulli's equation along a streamline

$$\frac{q^2}{2} + \frac{c^2}{\gamma - 1} = \frac{q^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{q_M^2}{2} \quad (11)$$

it follows that

$$\left(1 - \frac{u^2}{c^2}\right)\psi_{xx} - \frac{2uv}{c^2}\psi_{xy} + \left(1 - \frac{v^2}{c^2}\right)\psi_{yy} = \frac{-1}{2\gamma} \frac{G'(\psi)}{G(\psi)} \rho^2 q_M^2 \left(1 + \frac{q^2}{q_M^2}\right) \quad (12)$$

where

$$q^2 = u^2 + v^2 \quad (13)$$

$$c^2 = \gamma p / \rho \quad (14)$$

and  $q_M$  is the maximum velocity of the flow. This equation is hereafter referred to as the fundamental equation for compressible flow after the curved stationary shock wave given by equation (1). In it, the quantities  $u$ ,  $v$ ,  $c$ , and  $\rho$  are known functions of  $\psi$ ,  $\psi_x$ , and  $\psi_y$  determined by equations (9), (4), and (11).

It should be noted that equation (12) is equivalent to Crocco's equation (reference 5) for vortex motion. In fact, one is immediately derived from the other by the relation between the stream function defined above and that defined by Crocco.

Shock conditions expressed in terms of derivatives of stream function.— Let  $s$  and  $n$  be, respectively, the tangential and the normal directions at a point on the shock wave. Then

$$\left. \begin{aligned} \psi_x &= \frac{\partial \psi}{\partial s} \cos \tau + \frac{\partial \psi}{\partial n} \sin \tau \\ \psi_y &= \frac{\partial \psi}{\partial s} \sin \tau - \frac{\partial \psi}{\partial n} \cos \tau \end{aligned} \right\} \quad (15)$$

From continuity of the flow across the shock it follows that

$$\left. \begin{aligned} \frac{\partial \psi}{\partial s} &= \rho q_n = \rho_1 U \sin \tau \\ \frac{\partial \psi}{\partial n} &= -\rho q_t = -\rho U \cos \tau \end{aligned} \right\} \quad (16)$$



where  $q_t$  and  $q_n$  are, respectively, the tangential and the normal components of the velocity of the flow immediately after the shock wave. Thus, by virtue of equations (15) and (6), the shock conditions in terms of the derivatives of the stream function are transformed into the following form:

$$\left. \begin{aligned} \psi_x &= \rho_1 U \frac{F'(\rho_1 U y)}{1 + (F')^2(\rho_1 U y)} \frac{1 + (F')^2(\rho_1 U y) - M_1^2}{1 + (F')^2(\rho_1 U y) + \frac{\gamma - 1}{2} M_1^2} \\ \psi_y &= \rho_1 U - \rho_1 U \frac{(F')^2(\rho_1 U y)}{1 + (F')^2(\rho_1 U y)} \frac{1 + (F')^2(\rho_1 U y) - M_1^2}{1 + (F')^2(\rho_1 U y) + \frac{\gamma - 1}{2} M_1^2} \end{aligned} \right\} \quad (17)$$

Mathematical formulation of problem.— The problem of finding the two-dimensional-flow pattern after a stationary shock wave given by equation (1) is then to find the solution of the fundamental equation (12) with the conditions (17) on the shock wave. It is therefore a Cauchy initial-value problem if the shock wave is given.

However, it should be noted that the stream function  $\psi(x, y)$  thus obtained is, in general, many-valued if it is extended continuously (in physical variables) into the whole region after the shock wave. In the actual case, it is expected that other shock waves are formed near the tail of the body. Therefore the solution, once obtained, is limited by such shocks. The flow farther downstream should be treated separately. In fact, the flow after the shock wave can be treated separately for the subsonic region and the supersonic region.

Determination of physical variables.— After  $\psi(x, y)$  is found, the density distribution may be obtained from equations (4) and (11). The velocity components may then be gotten from equation (9). It should be remarked that this determination is two-valued: One value of the density corresponds to supersonic flow and the other, to subsonic flow. The location of the sonic line is therefore essential. When a solution is obtained for  $\psi$  (whether exact or approximate) the sonic line is immediately given by the condition that the two roots in Bernoulli's equation for the density  $\rho$  are equal. This gives

$$\left(\frac{\rho}{\rho_1}\right)^{\gamma-1} = \frac{\gamma-1}{\gamma+1} \left( \frac{2}{\gamma+1} + M_1^2 \right) \frac{1}{G(\psi)} = K_1 \frac{1}{G(\psi)}$$

Substituting back into Bernoulli's equation, it follows that

$$\frac{\psi_x^2 + \psi_y^2}{2} = K_2 [G(\psi)]^{\frac{-2}{\gamma-1}} \quad (18a)$$

where

$$K_2 = \frac{q_M^2}{2} (K_1)^{\frac{2}{\gamma-1}} - \frac{c_1^2 \rho_1^2}{\gamma-1} (K_1)^{\frac{\gamma+1}{\gamma-1}} \quad (18b)$$

and  $c_1$  is the free-stream velocity of sound.

In order that the sonic line thus determined should pass through the sonic point immediately after the shock wave,  $\psi_x^2 + \psi_y^2$  must satisfy the exact shock condition on the shock wave.

Approximations.— In the following discussions, restrictions are put on the shock shape such that

(1) The shock has its nose at  $(x_0, 0)$  where the shock is normal; that is,  $F'(0) = 0$ .

(2)  $F(y)$  is analytic in the real variable  $y$ ; moreover,  $F''(y)$  is positive.

(3) The shock wave tends to a Mach line asymptotically.

From the first two assumptions it follows that  $F'(y)$  is a monotonic function of  $y$  and  $yF'(y) \geq 0$ . From the last assumption, it is seen readily that  $0 \leq F'(y) \leq (M_1^2 - 1)^{1/2}$ .

The effect of vorticity generated by the shock can be better understood by examining the term in the right-hand side of the fundamental equation (12) for  $\psi$ . From equations (8), the term on the right-hand side of equation (12) is a product of  $F''(\psi) \rho^2 q_M^2$  and a nondimensional quantity. By a straightforward calculation this nondimensional factor is

$$\frac{F'(\psi)}{1 + (F')^2} \left[ 1 - \frac{(F')^2}{M_1^2 - 1} \right]^2 \left( 1 + \frac{q^2}{q_M^2} \right) \frac{(M_1^2 - 1)^2}{\gamma \left[ M_1^2 - \frac{1 + (F')^2}{2\gamma} (\gamma - 1) \right] \left[ M_1^2 + 2 \frac{1 + (F')^2}{\gamma - 1} \right]}$$

A rough estimation shows that it is bounded by

$$2 \frac{|F'(\psi)|}{(M_1^2 - 1)^{1/2}} \frac{1}{1 + (F')^2} \left[ 1 - \frac{(F')^2}{M_1^2 - 1} \right]^2 \frac{(M_1^2 - 1)^{5/2}}{\gamma \left( M_1^2 - \frac{\gamma - 1}{2\gamma} \right) \left( M_1^2 + \frac{2}{\gamma - 1} \right)}$$

It is therefore clear that, for the free-stream Mach number  $M_1$  close to 1, a perturbation process based on the parameter

$$(M_1^2 - 1)^{5/2} / \left( M_1^2 - \frac{\gamma - 1}{2\gamma} \right) \left( M_1^2 + \frac{2}{\gamma - 1} \right)$$

can be performed. The term in the right-hand side of equation (12) can be neglected for the first approximation. In fact, for any Mach number  $M_1$  in the general case,<sup>1</sup> the nondimensional factor is an infinitesimal of first order for small values of  $\psi$  and of second order for large values of  $\psi$ . The assumption of neglecting the effect of vorticity is perhaps reasonable for a certain range of  $M_1$  greater than unity.

In the following discussion the effect of vorticity will be neglected in finding approximate solutions for the stream function. The subsonic region will now be treated analytically by finding approximate solutions for the stream function. The exact shock conditions will be satisfied but the fundamental equation (12) will be satisfied only with the following approximations:

- (1) The effect of vorticity is neglected
- (2) Equation (12) is simplified by an approximate estimation of  $u/c$  and  $v/c$  as explicit functions  $\psi_x$  and  $\psi_y$

This estimation follows the same line as the one first suggested by Chaplygin (reference 6). The idea has been successfully applied by many authors (references 7 to 10) for analytic treatment of subsonic flows.

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<sup>1</sup>An upper bound for the nondimensional factor in terms of  $M_1$  only can be determined. The following is a trivial one:

$$(M_1^2 - 1)^2 / \gamma \left( M_1^2 - \frac{\gamma - 1}{2\gamma} \right) \left( M_1^2 + \frac{2}{\gamma - 1} \right)$$

It may further be suggested that the solution of the simplified equation for  $\psi$  satisfying the exact shock conditions may also serve as a first approximation for the supersonic region in the case of a thin body with a blunt nose. For such a case, the streamlines are likely to be nearly straight in the supersonic region. As the approximate solution of  $\psi$  obtained from the above considerations represents the true geometrical pattern of streamlines on the shock wave, far away from the body, and near to it, it may be expected to be a reasonable approximation in the whole region. However, this can be applied only if the two-dimensional body is an analytic curve. In case the body has a discontinuity, the flow downstream should be treated separately from the Mach line passing through this point.

After having obtained the first approximation (covering both the subsonic and the supersonic regions), an iteration procedure may then be carried out. This is done by evaluating all quantities in equation (12) except  $\psi_{xx}$ ,  $\psi_{xy}$ , and  $\psi_{yy}$  from the approximate solution and then solving the linear equation for  $\psi$ .

#### AN ANALYTIC APPROXIMATE SOLUTION IN THE LARGE FOR SUBSONIC REGION

Solution of simplified differential equation.— The solution of equation (12) satisfying the initial conditions (17) on the shock will now be approximately obtained by finding the stream function  $\psi(x,y)$  which satisfies the following simplified equation:

$$\left[ 1 - \left( \frac{\psi_y}{\rho_0 c_0} \right)^2 \right] \psi_{xx} + \frac{\psi_x \psi_y}{\rho_0^2 c_0^2} \psi_{xy} + \left[ 1 - \left( \frac{\psi_x}{\rho_0 c_0} \right)^2 \right] \psi_{yy} = 0 \quad (19)$$

with the same initial conditions (17). Equation (19) can be written as

$$\frac{\partial}{\partial x} \left[ \frac{\psi_x}{\sqrt{1 - (g^2/\rho_0^2 c_0^2)}} \right] + \frac{\partial}{\partial y} \left[ \frac{\psi_y}{\sqrt{1 - (g^2/\rho_0^2 c_0^2)}} \right] = 0 \quad (20)$$

where

$$g^2 = \psi_x^2 + \psi_y^2 \quad (21)$$

Equation (20) implies the existence of a function  $\phi$  such that

$$\left. \begin{aligned} \phi_x &= \frac{\psi_y}{\sqrt{1 - (g^2/\rho_o^2 c_o^2)}} \\ \phi_y &= - \frac{\psi_x}{\sqrt{1 - (g^2/\rho_o^2 c_o^2)}} \end{aligned} \right\} \quad (22)$$

It is now readily seen that the above development is completely equivalent to the Chaplygin theory. However, the usual physical interpretations of  $\phi$  and the partial derivatives of  $\phi$  and  $\psi$  are not implied. The formal relations, however, remain unchanged.

It is well-known that the function  $\phi$  satisfies the equation for the minimal surface and that the solution for  $\phi$  and  $\psi$  can be expressed as

$$\psi - i\phi = 2\rho_o c_o h(\Omega) \quad (23)$$

$$dz = dx + i dy$$

$$= \frac{dh}{\Omega} + \overline{\Omega} dh \quad (24)$$

where  $h(\Omega)$  is an arbitrary analytic function, and

$$\Omega = \frac{\psi_x - i\psi_y}{\sqrt{\rho_o^2 c_o^2 - g^2} + \rho_o c_o} \quad (25)$$

To satisfy the initial conditions (17), it is found convenient to apply the theory for minimal surfaces developed by Weierstrass, Björling,

and Schwarz (see reference 11). The final solution may be given in the following differential form:

$$\begin{aligned}
 dx = \operatorname{Re} \left\{ F'(\rho_1 U P) + 1 \frac{(1 - k^2) + k^2 \frac{(F')^2}{1 + (F')^2} \frac{1 + (F')^2 - M_1^2}{1 + (F')^2 + \frac{\gamma - 1}{2} M_1^2}}{\sqrt{(1 - k^2) + k^2 \frac{(F')^2}{1 + (F')^2} \frac{[1 + (F')^2 - M_1^2][1 + (F')^2 + \gamma M_1^2]}{[1 + (F')^2 + \frac{\gamma - 1}{2} M_1^2]^2}}} \right\} dP \\
 dy = \operatorname{Re} \left\{ 1 + i \frac{F' \left[ -1 + k^2 \frac{1}{1 + (F')^2} \frac{1 + (F')^2 - M_1^2}{1 + (F')^2 + \frac{\gamma - 1}{2} M_1^2} \right]}{\sqrt{(1 - k^2) + k^2 \frac{(F')^2}{1 + (F')^2} \frac{[1 + (F')^2 - M_1^2][1 + (F')^2 + \gamma M_1^2]}{[1 + (F')^2 + \frac{\gamma - 1}{2} M_1^2]^2}}} \right\} dP \\
 \frac{d\phi + i d\psi}{\rho_1 U} = \left\{ \frac{\frac{\gamma + 1}{2} M_1^2 F'}{1 + (F')^2 + \frac{\gamma - 1}{2} M_1^2} + 1 \frac{1}{\sqrt{(1 - k^2) + k^2 \frac{(F')^2}{1 + (F')^2} \frac{[1 + (F')^2 - M_1^2][1 + (F')^2 + \gamma M_1^2]}{[1 + (F')^2 + \frac{\gamma - 1}{2} M_1^2]^2}}} \right\} dP
 \end{aligned} \quad (26)$$

where  $\operatorname{Re}$  denotes the real part of a complex quantity,

$$k = \rho_1 U / \rho_0 c_0$$

and

$$F' = F'(\rho_1 U P)$$

and  $P$  is a complex running variable. It may be noted that this solution involves only the function  $F'(\rho_1 U y)$  for the slope of the shock wave.

Weierstrass' formula for minimal surfaces.- Weierstrass' formula for minimal surfaces (reference 11) reads as follows:

$$\left. \begin{aligned} x &= \operatorname{Re} \left[ i \left( f_1 - t f_1' - \frac{1-t^2}{2} f_1'' \right) \right] \\ y &= \operatorname{Im} \left[ i \left( f_1 - t f_1' + \frac{1+t^2}{2} f_1'' \right) \right] \\ w &= \operatorname{Re} \left[ i \left( -f_1' + t f_1'' \right) \right] \end{aligned} \right\} \quad (27)$$

where  $f_1$  is an analytic function of the complex variable  $t$ .

Now

$$t \frac{1}{2} f_1'''(t) dt = dh(1/it) \quad (28)$$

From equation (27) the following relation is immediately derived:

$$dz = dx + i dy = it dh(1/it) + \overline{dh(1/it)/it}$$

Since  $dw = \operatorname{Re} it f_1'''(t) dt = \operatorname{Re} 2i dh(\Omega) = d\phi/\rho_o c_o$ , the above relation reduces to equation (24) if one sets  $\Omega = 1/it$ .

Shock conditions and Björling problem.- As far as differential equation (19) is concerned, the parametric function  $h(\Omega)$  in expression (24) always generates an exact solution. There are still shock conditions to be satisfied. These conditions (17) raise the problem of finding the solution in the large of the Cauchy initial-value problem for minimal surfaces. Geometrically, it is now reduced to the problem of finding the minimal surface when a piece of space curve and the tangent plane at every point on the curve are prescribed. The curve is given in  $\vec{\xi}(x, y, w)$  space with the following parametric representation in vector form

$$\vec{\xi}_o = \left[ \frac{F(\rho_1 U y)}{\rho_1 U}, y, \phi/\rho_o c_o \right] \quad (29)$$

where

$$\begin{aligned}\phi/\rho_0 c_0 &= \phi \left[ \frac{F(\rho_1 U y)}{\rho_1 U}, y \right] \\ &= \int_0^y \left[ \phi_x F'(\rho_1 U y) + \phi_y \right] dy\end{aligned}$$

with  $\phi_x$  and  $\phi_y$  determined by equations (22) and (17). The direction of the normal of the tangent plane at a point  $\left[ F(\rho_1 U y)/\rho_1 U, y, \frac{\phi}{\rho_0 c_0} \right]$  is given by  $(\phi_x, \phi_y, -\rho_0 c_0)$ . This geometrical problem was first suggested by Björling in 1844 and solved by Schwarz in 1874.

Schwarz formula.— Let  $\vec{\xi}$  be a vector with components  $x, y,$  and  $w$ . Let  $\vec{\eta}(P)$  and  $\vec{\zeta}(Q)$  be complex-valued vector functions of the complex variables  $P$  and  $Q$ , respectively. The Schwarz formula for the Björling problem with the given curve  $\vec{\xi} = \vec{\xi}_0(r)$  and the normal of the tangent plane at any point on the curve  $\vec{k}(r)$  such that  $\vec{k}^2 = 1$  and  $\vec{k} \times \vec{\xi}_0' = 0$ , where  $\vec{\xi}_0'$  is the tangent vector of the curve, is (reference 11):

$$2\vec{\xi} = \vec{\eta}(P) + \vec{\zeta}(Q) \quad (30)$$

where

$$d\vec{\eta}(P) = d\vec{\xi}_0(P) + i[\vec{k}(P) \times d\vec{\xi}_0(P)] \quad (31)$$

$$d\vec{\zeta}(Q) = d\vec{\xi}_0(Q) - i[\vec{k}(Q) \times d\vec{\xi}_0(Q)] \quad (32)$$

From the last two relations it follows immediately that both vectors are isotropic lines; that is,

$$(\vec{\eta}')^2 = (\vec{\zeta}')^2 = 0$$



As the minimal surface should be real one has only to take  $Q$  as the complex conjugate of  $P$  and  $\zeta$ , as the complex conjugate vector function of  $\eta$ .

Exact solution of simplified differential equation satisfying shock conditions.— Now since

$$\vec{\kappa} = \left[ \psi_y / \rho_0 c_0, -\psi_x / \rho_0 c_0, -\sqrt{1 - (g^2 / \rho_0^2 c_0^2)} \right]$$

and  $\psi_x F' + \psi_y = \rho_1 U$  on the shock wave, it follows that

$$\begin{aligned} \vec{\kappa} \times d\vec{\xi}_0 &= \left[ \frac{1 - \rho_1 U \psi_y / \rho_0^2 c_0^2}{\sqrt{1 - (g^2 / \rho_0^2 c_0^2)}}, \frac{-F' + \rho_1 U \psi_x / \rho_0^2 c_0^2}{\sqrt{1 - (g^2 / \rho_0^2 c_0^2)}}, \frac{\rho_1 U}{\rho_0 c_0} \right] dy \\ &= \pi(y) dy \end{aligned} \quad (33)$$

where  $\psi_x$  and  $\psi_y$  are assigned values according to equation (17).

Let the components of this vector be denoted by  $[\pi_1(y), \pi_2(y), \pi_3(y)] dy$ .

If  $F(\rho_1 U y)$  is an analytic function of  $y$  and possesses an analytic extension in a domain in the  $\sigma$ -plane, where  $\sigma = y + i\omega$ , then equation (30) is the differential form

$$2 d\vec{\xi} = \text{Re} \left[ d\vec{\xi}_0(P) + i\pi(P) \right] dP \quad (34)$$

Or,

$$dx = \text{Re} \left[ F'(\rho_1 U P) + i \frac{1 - \rho_1 U \psi_y / \rho_0^2 c_0^2}{\sqrt{1 - (g^2 / \rho_0^2 c_0^2)}} \right] dP \quad (35)$$

$$dy = \operatorname{Re} \left[ 1 + i \frac{-F'(\rho_1 U P) + \rho_1 U \psi_x / \rho_o^2 c_o^2}{\sqrt{1 - (g^2 / \rho_o^2 c_o^2)}} \right] dP \quad (36)$$

$$d\phi = \operatorname{Re} \left[ \frac{\psi_y F'(\rho_1 U P) - \psi_x}{\sqrt{1 - (g^2 / \rho_o^2 c_o^2)}} + i \rho_1 U \right] dP \quad (37)$$

From the definition of  $h(\Omega)$ , by equation (23), and relation (28), it is evident that the last equation in equations (27) can also be written as

$$\frac{d\phi + i d\psi}{\rho_o c_o} = i t f'''(t) dt \quad (38)$$

Because of the fact that  $\eta(P)$  in equation (31) is an isotropic line when  $\bar{P} = \text{Constant}$ , or  $(\eta_1')^2(P) + (\eta_2')^2(P) + (\eta_3')^2(P) = 0$ , one can find an analytic relation which transforms  $t$  into  $P$  so that

$$\eta_1'(P) dP = i \frac{t^2 - 1}{2} f_1'''(t) dt \quad (39)$$

$$\eta_2'(P) dP = \frac{t^2 + 1}{2} f_1'''(t) dt \quad (40)$$

and

$$\eta_3'(P) dP = i t f_1'''(t) dt \quad (41)$$

Therefore equation (37) can also be written as the following:

$$d\phi + i d\psi = \left\{ \left[ \psi_y F'(\rho_1 U P) - \psi_x \right] / \sqrt{1 - (g^2 / \rho_o^2 c_o^2)} + i \rho_1 U \right\} dP \quad (42)$$

The final solution (26) is now obtained by substituting shock conditions (17) into equations (35), (36), and (42).

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